2.3 - Finding Limits

As we used the 1st and 2nd derivatives of a function f to determine its rate of change (direction) and concavity, here we are interested in the behavior of f near a fixed value of x or at infinity (asymptotic behavior) - Limits

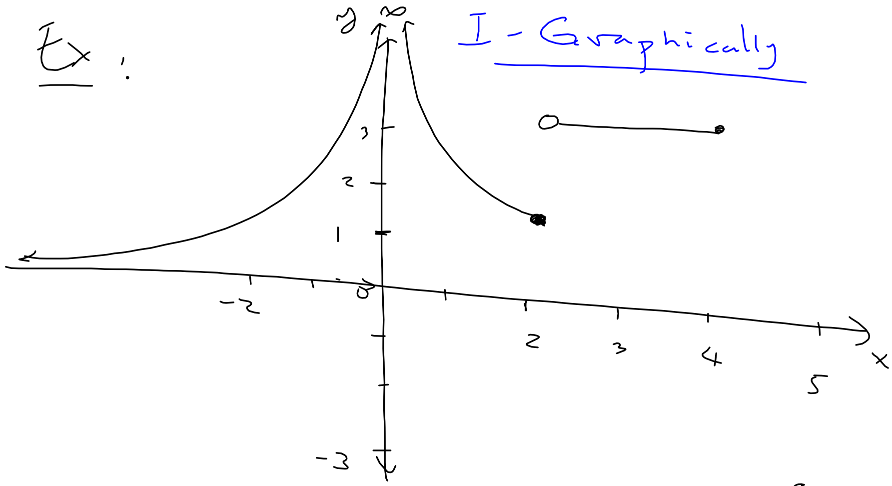
The limit of a function f near a value of x = c is written as $\lim_{x \to c} f(x)$ and read " Limit of f as x approaches c "

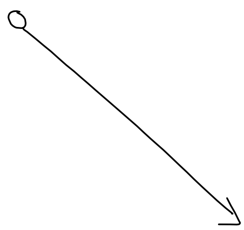
$$\lim_{x \to c} f(x) = \begin{cases} L & \text{if } \underbrace{\lim_{x \to c^-} f(x)}_{\text{(left side)}} = L = \underbrace{\lim_{x \to c^+} f(x)}_{\text{(right side)}} \\ \text{DNE} & \text{otherwise} \end{cases}$$

Ex :

I - Graphically

Consider the graph of f





$\lim_{x \to 0^-} f(x) = +\infty$; $\lim_{x \to 0^+} f(x) = +\infty$

"x approaches 0 from the left" "x approaches 0 from the right"

Conclusion: : $\lim_{x \to 0} f(x) = \infty$

Q: $\lim_{x \to 2} f(x) = ?$

(left side) $\lim_{x \to 2^-} f(x) = 1$

(right side) $\lim_{x \to 2^+} f(x) = 3$

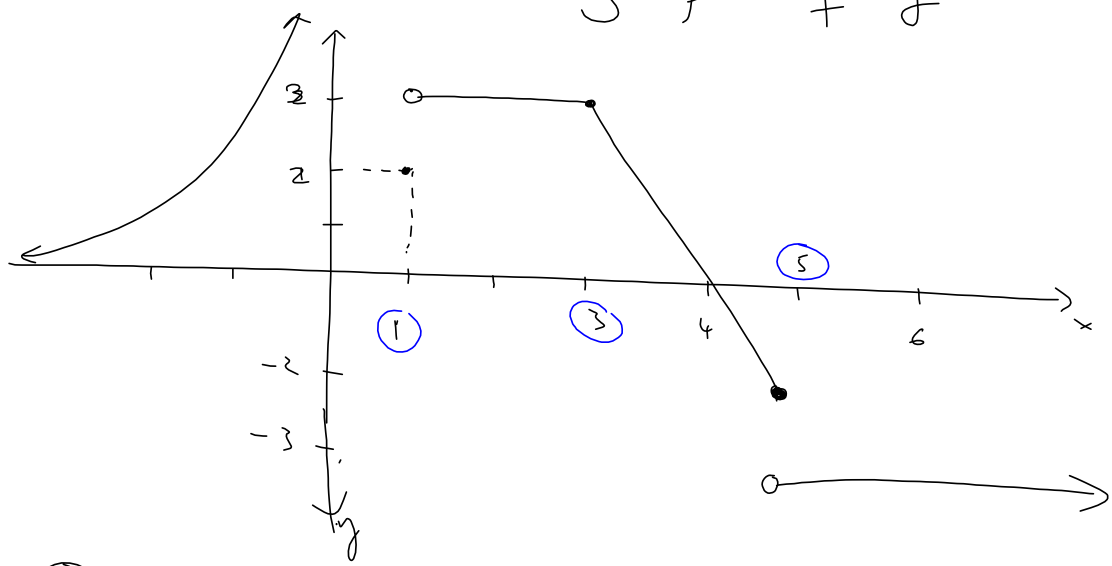
b/c $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

Conclusion: $\lim_{x \to 2} f(x) = \text{DNE}$

Likewise, $\lim_{x \to 4^-} f(x) = 3 \neq -3 = \lim_{x \to 4^+} f(x)$

So $\lim_{x \to 4} f(x) = \text{DNE}$

Consider the graph of g



Q: $g(1) = 2$ (value of g at $x=1$ is "2")

Q: $\lim_{x \to 1} g(x) = ?$

$\lim_{x \to 1^-} g(x) = DNE$! * Note

We should not attempt to find the limit at the left of "1" since g is Not defined on the left side of "1"

$\lim_{x \to 1^+} g(x) = 3$

b/c $\lim_{x \to 1^+} g(x) = 3 \neq 2 = g(1)$, we can say that g is Not Continuous at $x=1$

Q: Is g continuous at $x = 3$?

$\lim_{x \to 3^-} g(x) = 3$; $\lim_{x \to 3^+} g(x) = 3$ so

$\lim_{x \to 3} g(x) = 3$

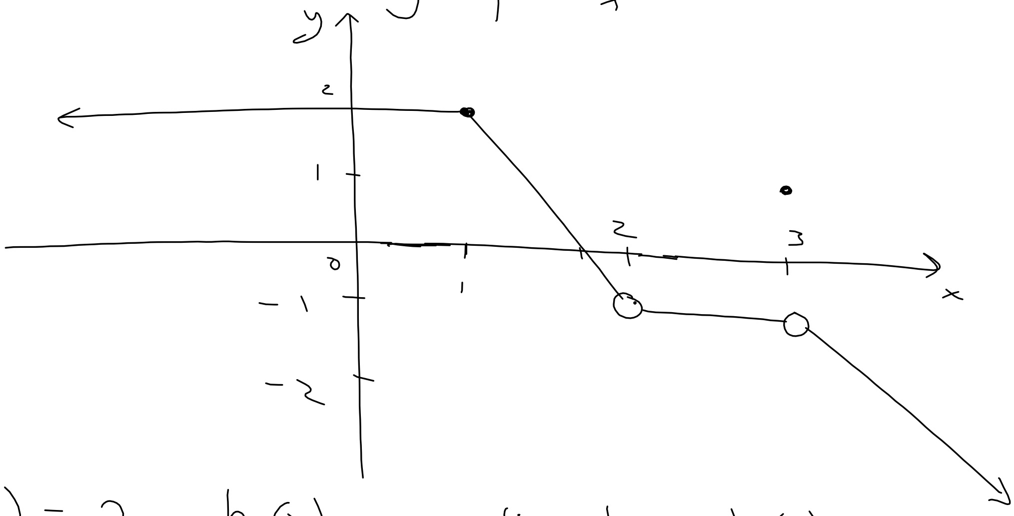
Now $g(3) = \underline{3} \Longrightarrow g$ is continuous at $\underline{x = 3}$

Formerly, suppose a function f is defined over an open interval containing c. If

$\lim_{x \to c} f(x) = L = f(c)$ then we say that

f is continuous at $x = c$. Otherwise, f is discontinuous or not continuous at $x = c$

Consider the graph of h:



$h(1) = 2$ $h(2) = \text{undefined!}$ $h(3) = 1$

$\lim_{x\to 1} h = 2$ b/c $\lim_{x\to 1^-} h = 2 = \lim_{x\to 1^+} h$ ⓐ

$\lim_{x\to 2} h = -1$ b/c $\lim_{x\to 2^-} h = -1 = \lim_{x\to 2^+} h$ ⓑ

$\lim_{x\to 3} h = -1$ b/c $\lim_{x\to 3^-} h = -1 = \lim_{x\to 3^+} h$ ⓒ

Conclusion:

ⓐ $\lim_{x\to 1} h = 2 = h(1)$ so h is continuous at $x = 1$

ⓑ $h(2) =$ undefined (though $\lim_{x\to 2} h = -1$), h is discontinuous at $x = 2$

ⓒ $\lim_{x\to 3} h = -1 \neq 3 = h(3)$ So h is Not continuous at $x = 3$

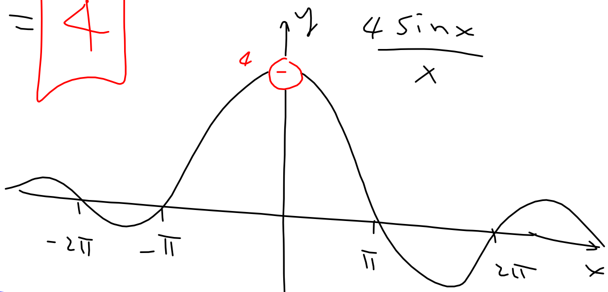
II - Computationally

Consider $f(x) = \frac{4\sin x}{x}$

Complete this table and estimate the limit of f as $x \to 0$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	3.99334	3.99993	4.00000	? undefined	4.00000	3.99993	3.99334

So $\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{4\sin x}{x}\right) = \boxed{4}$



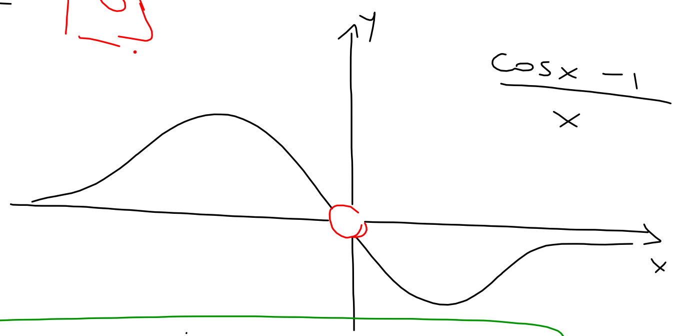
$$\boxed{\text{FACT: } \lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1}$$

Consider $f(x) = \dfrac{8\cos x - 8}{x}$

Complete the table and use it to estimate the limit of f as x approaches 0 (four decimals

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
f(x)	.3997	.0400	.0040	DNE	-0.0040	-.040	-.3997

$$\lim_{x \to 0} \left(\frac{8\cos x - 8}{x} \right) = \boxed{0}$$



$$\boxed{\underline{\text{FACT}}: \lim_{x \to 0} \left(\frac{\cos x - 1}{x} \right) = 0}$$